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SEMI-IMPLICIT RATIONAL RUNGE-KUTTA METHOD OF SOLVING SECOND ORDER DIFFERENTIAL EQUATION

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Abstract— There is much work done on implicit and explicit rational Runge-Kutta schemes for second order ordinary differential equations. This motivated the need for semi-implicit scheme since no work has been done on it for second order ordinary differential equations. The research was carried out using Taylor"s series and Binomial expansion after some related works were reviewed in the literature. The research was completed in which a one-stage semi-implicit rational Runge-Kutta scheme was derived, the convergence and consistency of the scheme was tested and the scheme was convergent and consistent. The adopted scheme was discovered to perform well in terms of approximating the exact solution of a special case of second order ordinary differential equations.

Keywords— **ordinary differential equations, Runge-Kutta, semi-implicit scheme**

I. INTRODUCTION

An ordinary differential equation (ODE) is an equation of the form:

$$y' = f(x,y) \qquad y(x_0) = y_0 \qquad a < x < b \qquad (1)$$

Arising from mathematical modeling of a large variety of physical problems in nuclear reactions, delay problems and computer aided designs, perturbation problems or dynamical processes in industries, technological fields and economy of some third world countries affected by inflation and other economic depressions. Often, the solution $y(x)$ of the ordinary differential equation (1) possesses components with fast and slow responses and one in which there is pole or discontinuities. The existing methods of solving this ordinary differential equation includes: Runge-Kutta method, Euler's method, implicit backward difference method proposed by Gear in 1971, Hybrid methods by Ademiluyi (1987), perturbed polynomial process by Lambert and Shaw, extrapolation process by Evans and Fatunla, Rational functions approximation method by Luke et al.

Runge-Kutta schemes are important family of implicit and explicit iterative methods for approximation of solution of ordinary differential equations. Consider the numerical approximation of second order initial value problems of the form:

$$y" = f(x,y,y'), \quad y(x_0) = y_0, \ y'(x_0) = y_0, \qquad a \le x \le b \qquad (1.1)$$

The general s− stage Runge-Kutta scheme for general second order initial value problems of ordinary differential equations of the form (1) as defined by Jain (1984) is:

$$y_{n+1} = y_n + hy'_n + \sum_{r=1}^{s} c_r k_r \qquad (2)$$

And

$$y'_{n+1} = y'_n + \frac{1}{h}\sum_{r=1}^{s} c'_r k_r \qquad (3)$$

Where

$$k_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{r} a_{ij}\, k_j, y'_n + \frac{1}{h}\sum_{j=1}^{\infty} b_{ij} k_j \right), \quad i(1)s \qquad (4)$$

With $c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j=1}^{i} b_{ij}, \quad i(1)r$ (5)

Where $c_i, a_{ij}, b_{ij}, c_r, c'_r$ are constants to be determined. The suitable parameter requires extremely lengthy algebraic manipulation, except for small values of s (Sharp et al, 1992 and Dormand et al, 1987).

In 1982, Hong Yuan Fu proposed a Rational Runge – Kutta scheme for the integration of this kind of ordinary differential equations. Okunbor (1985), Babatola and Ademiluyi (2000) and Bolaji (2005) worked further on this method and proposed schemes that can be classified as explicit, implicit or semi implicit method. Bolarinwa et al (2012) proposed a two-stage semi-implicit rational Runge-Kutta scheme for solving ordinary differential equation of first order.

From the above, the researcher will focus on developing a scheme for semi-implicit rational Runge-Kutta method of solving ordinary differential equation of second order.

II. Derivation of the Scheme

The rational form of (2.1) and (2.2) can be defined as

$$y_{n+1} = \frac{y_n + hy'_n + \sum_{i=1}^{r} w_i k_i}{1 + y_n \sum_{i=1}^{r} v_i H_i} \qquad (2.5)$$

$$y'_{n+1} = \frac{y'_n + \frac{1}{h}\sum_{i=1}^{r} w'_r k_i}{1 + \frac{1}{h} y'_n \sum_{i=1}^{r} v'_i H_i} \qquad (2.6)$$

where

$$k_i = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{i} a_{ij} k_j, y'_n + \frac{1}{h}\sum_{j=1}^{i} b_{ij} k_j \right), i = 1(1)r \qquad (2.7)$$

$$H_i = \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{i} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{i} \beta_{ij} H_j \right), i = 1(1)r \qquad (2.8)$$

with constraints

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j=1}^{i} b_{ij}, i = 1(1)r$$

$$d_i = \sum_{j=1}^{i} \alpha_{ij} = \frac{1}{2}\sum_{j=1}^{i} \beta_{ij}, i = 1(1)r$$

in which

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$$g(x_n, z_n) = -z_n^2 f(x_n, y_n) \text{and} z_n = \frac{1}{y_n}$$

The constraint equations are to ensure consistency of the method, h is the step size and the parameters $a_{ij}, b_{ij}, c_i, d_i, \propto_{ij}, \beta_{ij}$ are constants called the parameters of method.

Following Abhulimen and Uloku(2012), Bolarinwa(2012) procedures:

i. Obtain the taylor series expansion of k_i *and* H_i about the point (x_n, y_n, y'_n) and binomial series expansion of right-hand side of (2.7) and (2.8).
ii. Insert the Taylor series expansion into (2.7) and (2.8) respectively.
iii. Compare the final expansion of $k_i$andH_i about the point (x_n, y_n, y'_n) to the Taylor series expansion of y_{n+1}andy'_{n+1} about (x_n, y_n, y'_n) in the powers of h.

Normally the number of parameters exceed the number of equations, these parameters are chosen to ensure that (one or more of) the following conditions are satisfied.

i. Minimum bound of local truncation error exists
ii. The method has maximized interval of absolute stability
iii. Minimized computer storage facilities are utilized.

In equations (2.5),(2.6),(2.7) and (2.8), setting r=1 we have:

$$y_{n+1} = \frac{y_n + hy' + w_1 k_1}{1 + y_n v_1 H_1} \tag{2.9}$$

$$y'_{n+1} = \frac{y'_n + \frac{1}{h} w'_1 k_1}{1 + \frac{1}{h} y'_n v'_1 H_1} \tag{2.10}$$

where

$$k_i = \frac{h^2}{2} f\left(x_n + c_1 h, y_n + hc_1 y'_n + \sum_{j=1}^{1} a_{ij} k_j, y'_n + \frac{1}{h}\sum_{j=1}^{1} b_{ij}\, k_j\right), i = 1 \tag{2.11}$$

$$H_i = \frac{h^2}{2} g\left(x_n + d_1 h, z_n + hd_1 z'_n + \sum_{j=1}^{1} \propto_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{1} \beta_{ij}\, H_j\right), i = 1 \tag{2.12}$$

with constraints

$$c_1 = a_{11} = \frac{1}{2} b_{11}$$

$$d_1 = \propto_{11} = \frac{1}{2} \beta_{11}$$

since we are considering the semi-implicit scheme, $b_{ij} = 0$ for $j > i$.

Where $a_{ij}, b_{ij}, c_i, d_i, \propto_{ij}, \beta_{ij}, w_1, w_1', v_1, v_1'$ are all constants to be determined.

Now by adopting a binomial expansion (definition 1.6.10) on equations (2.9) gives

$$y_{n+1} = (y_n + hy'_n + w_1 k_1)(1 + y_n v_1 H_1)^{-1}$$

$$= (y_n + hy'_n + w_1 k_1)(1 - y_n v_1 H_1)$$

$$= y_n + hy'_n - (y_n^2 v_1 + hy_n y'_n v_1)H_1 + (w_1 - y_n v_1 H_1 w_1)k_1 \quad (2.13)$$

Similarly the binomial expansion of (2.10) gives

$$y'_{n+1} = \left(y'_n + \frac{1}{h} w_1' k_1\right)\left(1 + \frac{1}{h} y'_n v_1' H_1\right)^{-1}$$

$$= \left(y'_n + \frac{1}{h} w_1' k_1\right)\left(1 - \frac{1}{h} y'_n v_1' H_1\right)$$

$$= y'_n + \frac{1}{h} w_1' k_1 - \left(\frac{1}{h} {y_n'}^2 v_1' + \frac{1}{h^2} y_n' w_1' v_1' k_1\right) H_1 \quad (2.14)$$

Now, the Taylor's series expansion of y_{n+1} about x_n gives

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{iv} + \cdots$$

and

$$y_{n+1}' = y_n' + hy_n'' + \frac{h^2}{2!} y_n''' + \frac{h^3}{3!} y_n^{iv} + \cdots$$

where

$$y_n'' = f(x_n, y_n, y_n') = f_n$$

$$y_n''' = f_x + y' f_y + f_n f_{y'} = \Delta f_n$$

$$y_n^{iv} = f_{xx} + y_n'^2 f_{yy} + f^2 f_{y'y'} + 2y' f_n f_{yy'} + 2 f_n f_{xy'} + f_{y'} \Delta f_n = \Delta^2 f_n + f_{y'} \Delta f_n + f_n f_y$$

since $\Delta = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + f_n \frac{\partial}{\partial y'}$

Using the Taylor's series of the function of three variables (definition 1.6.8) we have from (2.11),

$$\frac{2}{h^2}k_1 = f_n + \left(c_1hf_x + (hc_1y_n' + a_{11}k_1)f_n + \frac{1}{h}b_{11}k_1f_{y'}\right)$$
$$+ \frac{1}{2!}\left[(c_1h)^2f_{xx} + 2c_1h(hc_1y_n' + a_{11}k_1)f_{xy} + 2c_1h\left(\frac{1}{h}b_{11}k_1\right)f_{xy'} + (hc_1y_n' + a_{11}k_1)^2f_{yy}\right.$$
$$\left. + 2(hc_1y_n' + a_{11}k_1)\left(\frac{1}{h}b_{11}k_1\right)f_{yy'} + \left(\frac{1}{h}b_{11}k_1\right)^2 f_{y'y'}\right] + \cdots$$

simplifying further and re-arranging the equation in powers of h gives

$$k_1 = \frac{h}{2}\left[b_{11}k_1f_{yy'} + a_{11}b_{11}k_1^2f_{yy'}\right] + \frac{h^2}{2}\left[f_n + a_{11}k_1f_y + c_1b_{11}k_1f_{xy'} + \frac{1}{2}a_{11}^2k_1^2f_{yy} \quad + c_1y_n'b_{11}k_1f_{yy'}\right]$$
$$+ \frac{h^3}{2}\left[c_1f_x + c_1y_n'f_y + c_1y_n'f_y + c_1a_{11}k_1f_{xy} + c_1a_{11}y_n'k_1f_{yy}\right]$$
$$+ \frac{h^4}{4}\left[c_1^2f_{xx} + 2c_1^2y_n'f_{xy} + c_1^2y_n'^2f_{yy}\right] + 0\left(h^3\right)(2.15)$$

Equation (2.15) is implicit, which cannot be proceeded by successive substitutions, we assume a solution of k_1 which may be expressed as

$$k_1 = hA_1 + h^2B_1 + h^3C_1 + h^4D_1 + O(h^3) \qquad (2.16)$$ in order to be able to compare and evaluate coefficients of *h*

Substituting the values of k_1 of (2.16) into (2.15) expand and re-arranging in powers of *h* gives

$$k_1 = \frac{h}{2}\left[b_{11}\left(hA_1 + h^2B_1 + h^3C_1\right)f_{y'} + a_{11}b_{11}\left(hA_1 + h^2B_1\right)^2f_{yy'}\right]$$
$$+ \frac{h^2}{2}\left[f_n + a_{11}\left(hA_1 + h^2B_1\right)f_y + C_1b_{11}\left(hA_1 + h^2B_1\right)f_{xy'} + a_{11}^2(hA_1)^2f_{yy}\right.$$
$$\left. + C_1y_n'b_{11}\left(hA_1 + h^2B_1\right)f_{yy'}\right] + \frac{h^3}{2}\left[c_1f_x + c_1y_n'f_y + c_1a_{11}(hA_1)f_{xy} + c_1a_{11}y_n'(hA_1)f_{yy}\right]$$
$$+ \frac{h^4}{4}\left[c_1^2f_{xx} + c_1^2y_n'f_{xy} + c_1^2y_n'^2f_{yy}\right] + 0\left(h^5\right)(2.17)$$

On equating the powers of h from (2.16) and (2.17) gives

$A_1 = 0$

$$B_1 = \frac{1}{2} f_n$$

$$C_1 = \frac{1}{2}\left[c_1 f_x + c_1 y_n' f_y + \frac{1}{2} b_{11} f_n f_{y'}\right] = \frac{1}{2} C_1 \Delta f_n$$

$$D_1 = \frac{1}{4}[c_1^2 \Delta^2 f_n + b_{11} \Delta f_n f_{y'} + a_{11} f_n f_y] \quad (2.18)$$

Then

$$k_1 = \frac{h^2}{2} f_n + \frac{h^3}{2} C_1 \Delta f_n + \frac{h^4}{4}[c_1^2 \Delta^2 f_n + b_{11} \Delta f_n f_{y'} + a_{11} f_n f_y] \quad (2.19)$$

In a similar manner

$H_1 = h^2 M_1 + h^3 N_1 + h^4 R_1 + 0(h^3)(2.20)$

where

$$M_1 = \frac{1}{2} g_n$$

$$N_1 = \frac{1}{2} d_1 \Delta g_n$$

$$R_1 = \frac{1}{4}[d_1^2 \Delta^2 g_n + \beta_{11} \Delta g_n g_{z'} + \alpha_{11} g_n g_z] \quad (2.21)$$

and also

$$H_1 = \frac{h^2}{2} g_n + \frac{h^3}{2} d_1 \Delta g_n + \frac{h^4}{4}[d_1^2 \Delta^2 g_n + \beta_{11} \Delta g_n g_{z'} + \alpha_{11} g_n g_z] \quad (2.22)$$

Substituting 2.16 and 2.20 into 2.13 and 2.14

$$y_{n+1} = y_n + hy'_n - (y_n^2 v_1 + hy_n y'_n v_1)(h^2 M_1 + h^3 N_1 + h^4 R_1) + (w_1 - y_n v_1 H_1 w_1)(h^2 B_1 + h^3 C_1 + h^4 D_1)$$

Expanding the brackets and re-arranging in powers of h gives

$$y_{n+1} = y_n + hy_n' + h^2(w_1 B_1 - y_n^2 v_1 M_1) + h^3(w_1 C_1 - y^2 v_1 N_1 - y_n y_n' v_1 M_1) + 0(h^3)$$

also for

$$y'_{n+1} = y'_n + \frac{1}{h}w'_1(h^2B_1 + h^3C_1 + h^4D_1) - \left(\frac{1}{h}y_n'^2 v'_1 + \frac{1}{h^2}y'_n w'_1 v'_1(h^2B_1 + h^3C_1 + h^4D_1)\right)(h^2M_1 + h^3N_1 + h^4R_1)$$

Expanding the brackets and re-arranging in powers of h gives

$$y'_{n+1} = y'_n + h(w'_1B_1 - y_n'^2 v'_1 M_1) + h^2(w'_1C_1 - y_n'^2 v'_1 N_1 - y'_n w'_1 v'_1 B_1 M_1) + h^3(w'_1D_1 - y_n'^2 v'_1 R_1 - y'_n w'_1 v'_1 B_1 N_1 - y'_n w'_1 v'_1 C_1 M_1) + 0(h^3)$$

Comparing the corresponding powers in h

$$\frac{1}{2}w_1 f_n - \frac{1}{2}y_n^2 v_1 g_n = \frac{1}{2}f_n$$

$$\frac{1}{2}w_1 C_1 \Delta f_n - \frac{1}{2}y_n'^2 v_1 d_1 \Delta g_n - \frac{1}{2}y_n y'_n v_1 g_n = \frac{1}{6}\Delta f_n$$

$$\frac{1}{2}w'_1 f_n - \frac{1}{2}y_n'^2 y'_n g_n = f_n$$

$$\frac{1}{2}w'_1 \Delta f_n - \frac{1}{2}y_n'^2 v'_1 d_1 \Delta g_n - \frac{1}{2}y'_n w'_1 v'_1 f_n\left(\frac{1}{2}g_n\right) = \frac{1}{2}\Delta f_n \qquad (2.23)$$

By using equation

$$g_n = -\frac{f_n}{y_n^2},\ g_x = -\frac{f_x}{y_n^2}g_z = -2\frac{f_n}{y_n} + f_y,\quad g_{z'} = -2\frac{f_n}{y_n} + f_{y'},\quad z'_n = -\frac{y'_n}{y_n^2} \qquad (2.24)$$

and

$$\Delta g_n = g_n + z'_n g_n + g_n g_{z'}$$

using 2.24 into 2.23 we get the following simultaneous equations

$$w_1 + v_1 = 1$$

$$w_1 c_1 + v_1 d_1 = \frac{1}{3}$$

$$w'_1 + v'_1 = 2$$

$$w'_1 c_1 + v'_1 d_1 = 1 \qquad (2.25)$$

Equation (2.25) has four (4) with six (6) unknowns; there will not be a unique solution, we choose d_1 to be zero in order to obtain a semi-implicit scheme. There will be a family of one-stage scheme.

Choosing the parameters

By solving (2.25) the following values were obtained so the equation 2.25 is satisfied and makes the scheme semi-implicit

$$w_1 = \frac{2}{3}, v_1 = \frac{1}{3}, c_1 = \frac{1}{2}, a_{11} = \frac{1}{2}, b_{11} = 1, d_1 = \alpha_{11} = \beta_{11} = 0,$$

$$w'_1 = 2, v'_1 = 0$$

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the scheme obtained is $y_{n+1} = \frac{y_n + hy_n^{'} + \frac{2}{3}k_1}{1+\frac{1}{3}y_n H_1}$ (2.26)

and

$$y_{n+1}^{'} = y_n^{'} + \frac{2}{h}k_1 \qquad (2.27)$$

where

$$k_1 = \frac{h^2}{2} f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hy_n^{'} + \frac{1}{2}k_1, y_n^{'} + \frac{1}{h}k_1\right) \qquad (2.27a)$$

$$H_1 = \frac{h^2}{2} g(x_n, z_n, z_n^{'}) \qquad (2.27b)$$

The schemes are semi-implicit as required, since k_1 is implicit and H_1 is explicit.

Choosing the parameters

$$w_1 = \frac{1}{3}, v_1 = \frac{2}{3}, c_1 = a_{11} = \frac{1}{2}, b_{11} = 1, since\ c_1 = a_{11} = \frac{1}{2}b_{11},\ w_1^{'} = 2, v_1^{'} = 0, d_1 = \alpha_{11} = \frac{1}{4}, but\ d_1 = \alpha_{11} = \frac{1}{2}\beta_{11} => \beta_{11} = \frac{1}{2}$$

By solving (2.25) the following values were obtained so the equation 2.25 is satisfied and makes the scheme semi-implicit

The following scheme is obtained

$$y_{n+1} = \frac{y_n + hy_n^{'} + \frac{1}{3}k_1}{1 + \frac{2}{3}y_n H_1}$$

and

$$y_{n+1}^{'} = y_n^{'} + \frac{2}{h}k_1$$

where

$$k_1 = \frac{h^2}{2}\ f\left(x_n + \frac{1}{2}h, y_n + \frac{h}{2}y_n^{'} + \frac{1}{2}k_1, y_n^{'} + \frac{1}{h}k_1\right)$$

$$H_1 = \frac{h^2}{2}\ g\left(x_n + \frac{1}{4}h, z_n + \frac{h}{4}z_n^{'} + \frac{1}{4}H_1, z_n^{'} + \frac{1}{2h}H_1\right)$$

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III. NUMERICAL PROBLEMS

Example 1.

Solve $y'' + 2y = 0, y(0) = 1, y'(0) = 0,$

The exact solution is given by

$y(x) = cosx\sqrt{2}, \quad y'(x) = -\sqrt{2}\, sinx\sqrt{2}$

Example 2.

Consider the equation $y'' = (1 + x^2)y, y(0) = 1, y'(0) = 0, x\epsilon[0,0.1]$

The exact solution is

$y(x) = e^{\frac{x^2}{2}}$

IV. DISCUSSION OF RESULTS

The first example 1 is an initial valued second order differential equation. This equation suits the scheme (3.26) as it is a second order ordinary differential equation of the form f(y) that is it is an autonomous system. The results get better with decrease in step size (h) as shown in table 1 in appendix A. 24
The table 2 in appendix A is the result obtained by applying the schemes (3.26) to example 2 evaluated from step size h=0.01, as it is a special case of second order ordinary differential equation of the form f(x,y). The result shows that the scheme performed well as the approximate solution approaches the exact solution better as the step size reduces to h=0.001 this implies the smaller the value of the step size, the better the approximated solution. This scheme approximates the special second order differential equation better as it is shown in table 2 in appendix A.

V. CONCLUSION

The semi-implicit rational Runge-Kutta scheme obtained in this research performed well in approximating the exact solutions and the result of the approximated solution gets better as the step size tends to zero, as it is shown in the result from table 2 in appendix A. The scheme is convergent as well as it is also consistent. This result implies that the scheme does better approximation on the special second order differential equation.

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APPENDIX A

Table 1: Solution of Example 1 at h=0.001

	Exact	solutions	Approximated	solutions	Error	
H	y(x)	y'(x)	y(x)	y'(x)	y	y'
0.001	0.99001665	-0.19933400	1.00000000	-0.00199999	9.98E-03	1.97E-01
0.002	0.96026596	-0.39468800	1.00000000	-0.00399999	3.97E-02	3.91E-01
0.003	0.91134193	-0.58216130	1.00000000	-0.00599997	8.87E-02	5.76E-01
0.004	0.84422141	-0.75801080	1.00000000	-0.00799997	1.56E-01	7.5E-01
0.005	0.76024460	-0.91872540	1.00000000	-0.00998752	2.40E-01	9.09E-01
0.006	0.66108821	-1.06109600	1.00000000	-0.01199998	3.39E-01	1.05E00
0.007	0.54873208	-1.18228010	1.00000000	-0.01399966	4.51E-01	1.17E00
0.008	0.42541959	-1.27985800	1.00000000	-0.01599949	5.75E-01	1.26E00
0.009	0.29361288	-1.35188130	1.00000000	-0.01799927	7.06E-01	1.33E00
0.010	0.99001665	-0.19933400	1.00000000	-0.01999999	9.98E-03	1.79E-01
0.020	0.96026596	-0.39468800	1.00000000	-0.03999200	3.97E-02	3.55E-01
0.030	0.91134193	-0.58216130	1.00000000	-0.05997302	8.87E-02	5.22E-01
0.040	0.84422141	-0.75801080	1.00000010	-0.07993605	1.56E-01	6.78E-01
0.050	0.76024460	-0.91872540	1.00000020	-0.09987515	2.40E-01	8.19E-01
0.060	0.66108821	-1.06109600	1.00000040	-0.11978440	3.39E-01	9.41E-01
0.070	0.54873208	-1.18228010	1.00000080	-0.13965780	4.51E-01	1.04E00
0.080	0.42541959	-1.27985800	1.00000140	-0.15948960	5.75E-01	1.12E00
0.090	0.29361288	-1.35188130	1.00000220	-0.17927390	7.06E-01	1.17E00
0.100	0.15594369	-1.39691200	1.00000330	-0.19900490	8.44E-01	1.20E00
0.200	0.96026596	-0.39468800	1.00053700	-0.39215690	4.02E-02	2.53E-03
0.300	0.91134193	-0.58216130	1.00274900	-0.57416270	9.14E-02	7.80E-03
0.400	0.84422141	-0.75801080	1.00884500	-0.74074070	1.65E-01	1.73E-02
0.500	0.76024460	-0.91872540	1.02222200	-0.88888890	2.62E-01	2.98E-02
0.600	0.66108821	-1.06109600	1.04817100	-1.01694900	3.87E-01	4.41E-02
0.700	0.54873208	-1.18228010	1.09547100	-1.12449800	5.47E-01	5.78E-02
0.800	0.42541959	-1.27985800	1.18040900	-1.21212100	7.55E-01	6.77E-02
0.900	0.29361288	-1.35188130	1.33838800	-1.28113900	1.04E00	7.07E-02
1.000	0.15594369	-1.39691200	1.66666700	-1.33333300	1.51E00	6.36E-02

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Table 2: Solution to Example 2 at h=0.001

	Exact	Solutions	Approximated	Solutions	Error	
H	y(x)	y'(x)	y(x)	y'(x)	y	y'
0.001	1.00000050	0.00100000	1.00000000	0.00100000	5.00E-07	0.00E00
0.002	1.00000200	0.00200000	1.00000100	0.00200000	1.00E-06	0.00E00
0.003	1.00000450	0.00300001	1.00000100	0.00300000	3.50E-06	1.00E-09
0.004	1.00000800	0.00400003	1.00000300	0.00400003	5.00E-06	0.00E00
0.005	1.00001250	0.00500006	1.00000400	0.00500006	8.50E-06	0.00E00
0.006	1.00001800	0.00600011	1.00000600	0.00600011	1.20E-05	0.00E00
0.007	1.00002450	0.00700017	1.00000800	0.00700017	1.65E-05	0.00E00
0.008	1.00003200	0.00800026	1.00001100	0.00800026	2.10E-05	0.00E00
0.009	1.00004050	0.00900037	1.00001400	0.00900037	2.65E-05	0.00E00
0.010	1.00000050	0.00100000	1.00001700	0.01000050	1.65E-05	9.00E-03
0.020	1.00000200	0.00200000	1.00006700	0.02000400	6.50E-05	1.80E-02
0.030	1.00000450	0.00300001	1.00015000	0.03001351	1.46E-04	2.70E-02
0.040	1.00000800	0.00400003	1.00026700	0.04003202	2.60E-04	3.60E-02
0.050	1.00001250	0.00500006	1.00041800	0.05006256	4.06E-04	4.51E-02
0.060	1.00001800	0.00600011	1.00060200	0.06010815	5.84E-04	5.41E-02
0.070	1.00002450	0.00700017	1.00082000	0.07017181	7.96E-04	6.32E-02
0.080	1.00003200	0.00800026	1.00107200	0.08025661	1.04E-03	7.23E-02
0.090	1.00004050	0.00900037	1.00135900	0.09036561	1.32E-03	8.14E-02
0.100	1.00005000	0.01000050	1.00501900	0.10050190	4.97E-03	9.05E-02