International Journal of Engineering Applied Sciences and Technology, 2021
Vol. 5, Issue 10, ISSN No. 2455-2143, Pages 147-150
Published Online February 2021 in IJEAST (http://www.ijeast.com)

HAMILTONICITY OF CAMOUFLAGE GRAPHS

Sowmiya K
Assistant Professor, Department of Mathematics
Mangayarkarasi College of Arts and Science for Women, Madurai, Tamil Nadu, India

Abstract— **This paper examines the Hamiltonicity of graphs having some hidden behaviours of some other graphs in it. The well-known mathematician Barnette introduced the open conjecture which becomes a theorem by restricting our attention to the class of graphs which is 3-regular, 3-connected, bipartite, planar graphs having odd number of vertices in its partition be proved as a Hamiltonian. Consequently the result proved in this paper stated that "Every connected vertex-transitive simple graph has a Hamilton path" shows a significant improvement over the previous efforts by L.Babai and L.Lovasz who put forth this conjecture. And we characterize a graphic sequence d which is forcibly Hamiltonian if every simple graph with degree sequence d is Hamiltonian. Thus we discussed about the concealed graphs which are proven to be Hamiltonian.**

Keywords— **Barnette's graph, Vertex transitive graph, Multiple n-lines**

I. INTRODUCTION

The Most dominant branch of mathematics known as the Graph Theory widely applies in every field covering all areas being largely utilized by researchers to bring out their solution in the form of graphs, which are mathematical structure used to model the pairwise relations between the objects. Many real-world situations can conveniently be described by means of a diagram consisting of set of points together with the lines joining certain pairs of these points. There is a large utilization of graphs in mathematics provides the problem solutions. The graphs we discussed here are non-trivial and connected graphs.

II. DEFINITIONS

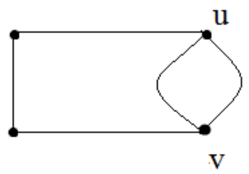
Graph: A graph is an ordered triple (V(G), E(G),ϕ(G)) consisting of a non-empty set of vertices V(G), a set E(G),disjoint from V(G), of edges and an incidence function ϕ(G) that associates with each edge of G. If e is an edge and u,v are vertices of G such that ϕ(e) = uv then e is said to join u and v; the vertices u and v are called the ends of e.

Camouflage graph: The Camouflage graph is a graph which has some hidden graphs in it. (i.e.) we can find one or more behaviours of some types of graphs in a single graph. Barnette's graph is an example for this graph.

Multigraph: If more than one line joining two vertices is allowed, the resulting object is called a multigraph.

Multiple n-lines: An edge which is incident with the same pair of vertices n-times is called multiple n-lines.

Example: Multiple 2-lines of a graph G, where an edge is adjacent to the vertices u and v, n times.



Graph G with Multiple 2-lines (u,v)

Regular graph: For any graph G, we define

$$\delta(G) = \min\{\deg v \ / v \in G\}$$
$$\Delta(G) = \max\{\deg v \ / v \in G\}$$

If all the points of G have the same degree r then $\delta(G) = \Delta(G) = r$ and in this case G is called a regular graph of degree r.

Cubic graph: A regular graph of degree 3 is called a cubic graph.

k-Connected graph: The vertex-connectivity $k(G)$ is the minimum size of a vertex set S such that $G - S$ is disconnected. If S consists of single vertex v, we call v a cut-vertex. We say that a graph G is k -connected if its connectivity is at least k.

Bipartite graph: A graph is said to be bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point of V_2. (V_1,V_2) is called the bipartition of G. If further every line of G joins a point of V_1 to the points of V_2 then G is called a bipartition of G.

Planar graph: A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends.

Vertex-transitive graph: A vertex-transitive graph is a graph whose automorphism group is transitive.

Hamilton path: A path that contains every vertex of G is called the Hamiltonian path of G.

Hamilton cycle: It is a cycle of G that contains every vertex of G.

Hamiltonian graph: A graph is Hamiltonian if it contains a Hamiltonian cycle, i.e. a cycle that contains every vertex exactly once.

III. A BRIEF STUDY ABOUT THE BARNETTE'S CONJECTURE

In 1884, the conjecture of Tait's stated that "Every cubic planar 3-connected graph is Hamiltonian". Later his research in this conjecture results in the new conjecture (1971) stated "Every cubic bipartite 3-connected graph is Hamiltonian". This was being proved by the counter examples of Baraev & Faradzhev (1978) and Horton (1982). In between these Barnette's put forth the Conjecture in 1969 stated that "Every cubic bipartite planar 3-connected graph is Hamiltonian. P.Goodey (1975) has verified this conjecture for plane graphs that contains only faces of degree four or six.

We now verify Barnette's conjecture for cubic 3-connected bipartite planar graph having equal number of odd vertices in each of its partition. The below theorem proves the conjecture with some restrictions in the vertex set of bipartite graph.

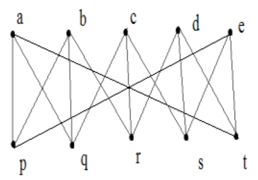
IV. THEOREM 1

We show that every cubic 3-connected bipartite planar graph G is Hamiltonian ⇔ there are odd number of vertices in each of the two partitions V_1 & V_2 of $V(G)$ and $V_1 = V_2$ such that there exists a vertex in V_1 joins twice to a vertex in V_2. (i.e.) there exists exactly one multiple 2- lines in the graph G. Proof: Assume that G is a cubic, 3-connected, bipartite and planar graph having a Hamiltonian cycle and there are odd numbers of vertices in each of the partition of $V(G)$ and having equal number of vertices in both the partition.

To prove that there exists a vertex in V_1 joins twice to a vertex in V_2. (i.e.) there exists exactly one multiple 2- lines in the graph.

Suppose if there is no such lines exist then the graph may of following consequences may arise:

(1) The graph G may be simple, but since G is cubic, every vertex in G must be adjacent to exactly three vertices. Since there are odd number of vertices in each partition, there must be an edge intersects the other edge while drawing it in a plane, so that the planarity fails. As evidence, counterexample is given in the following figures. A cubic 3-connected bipartite simple graph in figure1.1 is drawn on a plane with intersecting edges as in figure 1.2 showing the non-planarity.



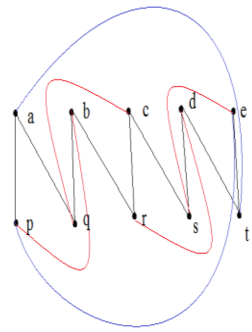


fig:1.1 cubic,3-connected, bipartite graph fig 1.2 cubic, 3-connected, bipartite and non-planar graph

(2) Suppose, if G has loops, then the graph cannot be 3-connected and not be a cubic graph. Since there is a vertex which has loop cannot be adjacent to more than one vertex other than itself. So there must be exactly two vertices with degree 2. The vertices represented as 'p' and 'r' in the example figure 1.3 which cannot have the adjacency from the vertex 'e'. Similarly if we have more than one loop then it will fails to satisfy the hypothesis because those vertices have loops contains cannot have the adjacency with the other vertices which does not have loops. Hence the graph G must not contain any loop.

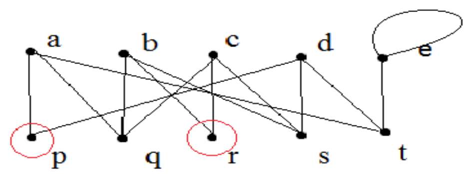


fig:1.3 non-cubic,1-connected bipartite graph

(3) Possibly, if there are two pairs of vertices that are adjacent to each other vertices twice, then if we draw it in the plane it become a non-planar graph having intersecting edges as shown in the figure 1.5.The edges (a,q) and (e,p) are multiple 2-edges which is drawn on a plane with intersecting edges results in non-planar graph.

International Journal of Engineering Applied Sciences and Technology, 2021
Vol. 5, Issue 10, ISSN No. 2455-2143, Pages 147-150
Published Online February 2021 in IJEAST (http://www.ijeast.com)

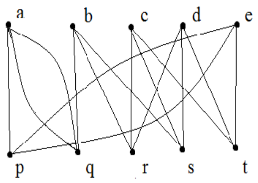


fig:1.4 Multigraph with two multiple 2-lines

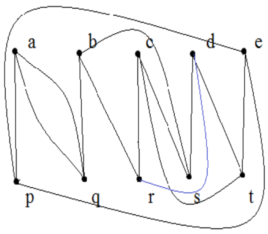


fig:1.5 3-connected, bipartite, non-planar graph with two multiple 2-lines

(4) If a graph G is pseudo graph, then it will not have $\delta(G) = \Delta(G)$. Thus the graph cannot be regular. Hence it is not possible that a graph to be pseudo graph as an example shown in the figure 1.6.

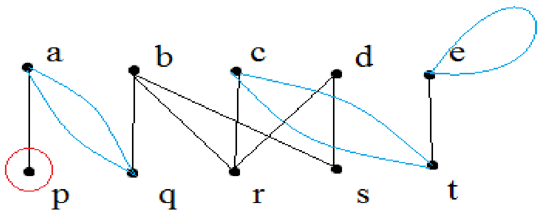


fig:1.6 pseudo graph

Thus the graph G should contains exactly one multiple 2-lines only then it would contain a Hamiltonian cycle, where G is 3-connected cubic, bipartite, planar graph having odd number of vertices in each of its partition. The following figures 1.6 &1.7 represent the Barnette graph.

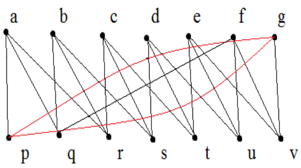


fig:1.7 cubic, 3-connected, bipartite, planar graph with one multiple 2-lines

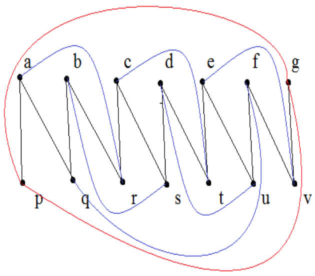


fig:1.8 Planar embedding of figure 1.7

Converse part is obtained by retracing the steps explained in the Implies part of the theorem.

V. HAMILTONICITY IN VERTEX TRANSITIVE GRAPHS

Vertex transitive graphs are very well behaved graphs, there is only five vertex transitive graphs which do not have Hamiltonian cycles and there are no graphs known which do not contain a Hamiltonian path. As a consequence Laszlo Lovasz gives the conjecture that "Every vertex transitive graph has a Hamiltonian path has verified by L.Babai for graphs with a prime number of vertices. Even though the problem was already investigated by the lots of people, here we workout the some results known and found some holes to close.

VI. THEOREM 2

Every connected vertex-transitive simple graph has a Hamilton path.

Proof: Let G be a simple vertex transitive connected graph with $V \geq 3$. We know that "If closure of G, $C(G)$ is complete, then G is Hamiltonian".

To prove that G has a Hamiltonian path, it is enough to prove that $C(G)$ $+e_1$is complete, where e_1 is an edge of the graph G.(i.e.) $C(G)$ is non-complete unless $e_1$is added to it. To prove $e_1 = uv$ is an edge of G. Here we denote the degree of a vertex v in $C(G)$ by $d(v)$.

Without loss of generality assume that $C(G)$ is not complete. Let u and v be the non-adjacent vertices in $C(G)$ with $d(u) \leq d(v)$ (2.1) and $d(u) + d(v)$ as large as possible. And also

$$d(u) + d(v) < V \quad (2.2)$$

because u and v are nonadjacent they cannot have degree sum V or more.

Now denote by Q the set of vertices in $V - \{v\}$ which are nonadjacent to v in $C(G)$ and by R the set of vertices in $V - \{u\}$ which are nonadjacent to u in $C(G)$

Clearly $|Q| = v - 1 - d(v)$ and $|R| = v - 1 - d(u)$. (2.3)

Furthermore by the choice of u and v, each vertex in Q has degree atmost $d(u)$ and each vertex in $R \cup \{u\}$ has at most $d(v)$.

Therefore by (2.3) there exists a vertex adjacent to u and also to v but since G is vertex transitive graph, there must be adjacency between u and v as well.

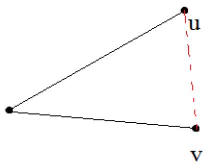


fig:2.1

Hence $e_1 = uv$ is an edge of G, thus $C(G)$ is complete and so G must contain a Hamilton path.

International Journal of Engineering Applied Sciences and Technology, 2021
Vol. 5, Issue 10, ISSN No. 2455-2143, Pages 147-150
Published Online February 2021 in IJEAST (http://www.ijeast.com)

VII. SOME CHARACTERIZATION OF FORCIBLY HAMILTONIAN DEGREE SEQUENCES:

Let the degree sequence of a graph with $p \geq 3$ points be $(d_1, d_2, d_3, \ldots\ldots d_p)$. It must be of non-decreasing sequence (i.e.) $d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_p$, each of value at least 2.

If the degree sequence $(d_1, d_2, d_3, \ldots\ldots d_p)$ has $d_1 = d_2 = d_3 = \ldots = d_p$ having the values at least 2, (see example figure 3.1).

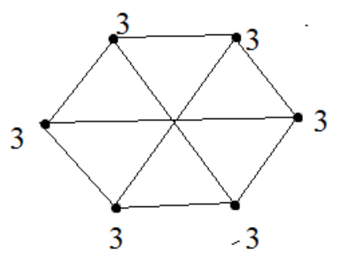


fig:3.1

For no $q < \frac{v}{2}$ there is $d_q < q$ and $d_{v-q} \leq v - q$, then the graph is Hamiltonian.

VIII. CONCLUSION

The fundamental motive of this paper is to present the significance of the graph theoretical thoughts in the emerging research field. Using Barnette's Conjecture several consequences were analyzed and that gives a new view for further researches.

IX. REFERENCE

[1] J.A.Bondy and U.S.R.Murty, Graph Theory with Applications (*MacMillan, New york 1976).*

[2] R. E. L. Aldred, S. Bau, D. A. Holton and B. D. McKay, "Nonhamiltonian 3-Connected Cubic Planar Graphs." *SIAM J. Disc. Math. 13, 25-32, 2000.*

[3] R. E. L. Aldred, S. Bau, D. A. Holton and B. D. McKay, Cycles through 23 vertices in 3-connected cubic planar graphs, Graphs *Combin. 15(4) (1999), 373– 376.*

[4] Narsingh Deo, Graph Theory with Applications to Engineering and Computer Science, (*PHI ,1974*).

[5] Bollobas, Bela Graph Theory, An Introductory Course

[6] John Adrian Bondy, Graph theory with applications

[7] Douglas & West, Introduction to graph theory

[8] Arthur T.Benjamin, Gary Chartand, and Ping Zhang ,The Fascinating World of Graph Theory

[9] Santanu Saha ray, Graph Theory with Algorithms and its Applications *(In Applied Science and Technology*)

[10] N.L. Biggs, E.K.Lloyd, R.J.WilsonGraph theory (*1736-1936*)

[11] Norman L.Biggs, Discrete Mathematics, Second Edition (*OXFORD*)

[12] Norman Biggs, Algebraic Graph Theory Second Edition, *Cambridge University Press, 1974, 1993.*

[13] Russell Merris, Graph theory, *A Wiley -Inter Science Publication , John Wiley & Sons, INC.*

[14] Gary Chartand, Linda Lesniak and Ping Zhang, Graphs and Digraphs *CPC Press,A Chapman & Hall Book.*

[15] Jonnathan L.Gross, Jay Yellen, Hand Book of Graph Theory

[16] R.Venkata Rao, Decision Making in the Manufacturing Environment Using Graph Theory and Fuzzy Multiple Attribute Decision Making,*British Library Cataloguing in /publication Data, Springer.*