AN ITERATIVE FORMULA FOR SIMULTANEOUS LOCATION OF THE ZEROS OF A POLYNOMIAL

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Abstract—Needless to say that the search for efficient algorithms for determining zeros of polynomials has been continually raised in many applications. In this paper we give a cubic iteration method for determining simultaneously all the zeros of a polynomial – assumed distinct – starting with ‘reasonably close’ initial approximations – also assumed distinct. The polynomial – in question – is expressed in its Taylor series expansion in terms of the initial approximations and their correction terms. A formula with cubic rate of convergence – based on retaining terms up to $2^{nd}$ order of the expansion in the correction terms – is derived.

I. INTRODUCTION

The problem of determining all the zeros of a polynomial simultaneously has been considered by many, nonetheless a lot more is still being sought.

Without loss of generality, let us consider monic polynomials i.e. polynomials with 1 as leading coefficient.

Let $P(z) = \prod_{i=1}^{n} (z - w_i)$ (1)

be such a polynomial with $w_i, i = 1,2,..,n$ - assumed distinct - as its zeros and $z_i, i = 1,2,..,n$ – also assumed distinct - as their approximations.

Rewriting (1) as:

$$P(z) = \prod_{i=1}^{n} (z - w_i) = \prod_{i=1}^{n} (z - z_i - \Delta_i)$$

Or in expanded form, we have:-

$$P(z) = \prod_{i=1}^{n} (z - z_i) - \sum_{i=1}^{n} \prod_{i=1}^{n} (z - z_i) + \sum_{i=1}^{n} \Delta_i \sum_{i=1}^{n} \prod_{i=1}^{n} (Z - z_i) + ...$$

(Higher Order Terms) (3)

Putting $z = z_i$ in Eq. (3), we have

$$P(z_i) = - \sum_{i=1}^{n} \Delta_i \prod_{i=1}^{n} (Z - z_i) + \sum_{i=1}^{n} \Delta_i \sum_{i=1}^{n} \prod_{i=1}^{n} (z - z_i) +$$

(4)

Defining $Q(z) = \prod_{i=1}^{n} (Z - z_i)$ (5)

and noting that $Q(z_i) = 0 \neq Q'(z_i) = \prod_{i=1}^{n} (z - z_i), r=1,2,..,n$ (6)

It can be established that :

$$\sum_{i=1}^{n} \Delta_i \prod_{i=1}^{n} (z - z_i) = \Delta \cdot Q'(z_i)$$

(7)

II. DERIVATION OF THE METHOD

On ignoring Higher Order Terms and from Eqs. (4) to (7), we can deduce :-

$$P(z) + \Delta \cdot Q'(z_i) - \Delta \cdot Q'(z_i) \sum_{i=1}^{n} \prod_{i=1}^{n} \prod_{i=1}^{n} (z - z_i)$$

(8)

$$Q'(z_i) = \prod_{i=1}^{n} (z - z_i)$$

(9)

Now, truncating Eq.(8) after the 1$^{st}$ order term we have

$$P(z) + \Delta \cdot Q'(z_i) = 0$$

(10)

Giving $\Delta = P(z_i) / Q'(z_i)$

(11)

hence-forth denoted by $\tilde{\delta_i} ( \approx - P(z_i) / Q'(z_i) )$, the expression given by Durand Kerner, known to give quadratic convergence.

Truncating Eq. (4) after the 2$^{nd}$ order term, we obtain Eq (8), which can be rearranged to give an expression for $\Delta_i$ - the theme of our method, namely:-

$$\Delta_i = - P(z_i) / Q'(z_i) \left[ 1 - \sum_{i=1}^{n} \prod_{i=1}^{n} \prod_{i=1}^{n} (z - z_i) \right]$$

(12)

For practical computational purposes and with $\tilde{\delta_i}$ ( $\approx - P(z_i) / Q'(z_i)$ ), this may be approximated and rephrased as :-

$$\Delta_i \approx \tilde{\delta_i} / \left[ 1 - \sum_{i=1}^{n} \prod_{i=1}^{n} \prod_{i=1}^{n} (z - z_i) \right]$$

(13)

III. CONCLUSION AND COMMENTS

The method is simple and easy to apply.

To understand and really comprehend the computational procedure and to have a feeling of the effectiveness of the method, appreciating its convergence rate, without loss of generality, it suffices to give an example of a cubic and confine our attention to finding the improvements to the initial crude approximations obtained via the first iteration cycle.
Further better improvements can be attained via executing the pattern - repeatedly - with the new updated *z*'s after the ∆'s have been incorporated in them.

IV. EXAMPLE

Consider the polynomial

\[ P(z) = z^3 - z^2 - 81z + 81, \]

with 10, -10 and 0 as crude approximations to its zeros : 9, -9 and 1.

<table>
<thead>
<tr>
<th>z</th>
<th>z1 = 10</th>
<th>z2 = -10</th>
<th>z3 = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(z1)</td>
<td>171</td>
<td>-209</td>
<td>81</td>
</tr>
<tr>
<td>Q'(z1)</td>
<td>200</td>
<td>200</td>
<td>-100</td>
</tr>
<tr>
<td>(\hat{e}_r)</td>
<td>-0.855</td>
<td>1.045</td>
<td>0.81</td>
</tr>
<tr>
<td>(\Delta_r)</td>
<td>-0.986</td>
<td>0.9705</td>
<td>0.999</td>
</tr>
</tbody>
</table>

Updating \(z_r\) by \(\Delta_r\) above \(r = 1, 2, 3\) in the light of the method we have

| z1 = 9.014 (9) | z2 = -9.0295 (-9) | z3 = 0.9994 (1) |

** numbers in () represent the actual zeros, quoted for comparison.

V. REFERENCES


